Non-identifiability of word embeddings, and connections to shape analysis

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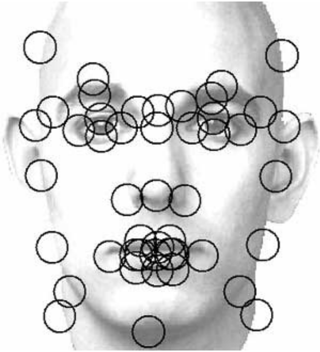
Shape and invariances



"Configuration"

$$\rightarrow \quad \overset{m\times k}{\mathbf{V}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \\ z_1 & z_2 & \cdots & z_k \end{pmatrix}$$

- "Shape": info invariant to translation, scaling, rotation (+ reflection)

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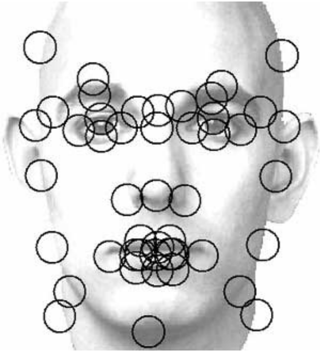

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(Assume $\mathbf{V}$ is centred: $\mathbf{V1} = \mathbf{0}$)

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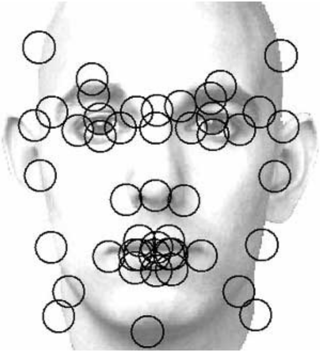

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- "Shape": info invariant to translation, scaling, rotation (+ reflection)
- Can identify shape as $[\mathbf{V}] = \{c\mathbf{Q}\mathbf{V} : c \in \mathbb{R}^+; \mathbf{Q} \in O(m)\}$

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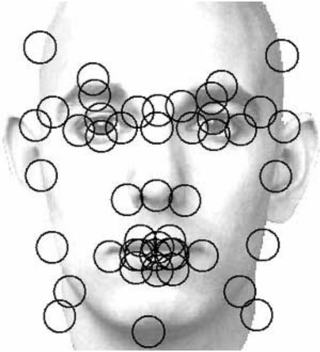

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(Assume $\mathbf{V}$ is centred: $\mathbf{V1} = \mathbf{0}$)

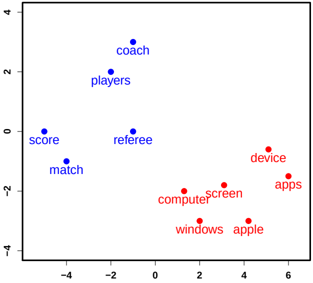
- "Shape": info invariant to translation, scaling, rotation (+ reflection)
- Can identify shape as $[\mathbf{V}] = \{c\mathbf{QV} : c \in \mathbb{R}^+; \mathbf{Q} \in O(m)\}$
- "Shape function": $g(\cdot)$ such that $g(\mathbf{V}) = g(c\mathbf{QV})$.

Text data

Representation, $\mathbf{X}$

$$
\begin{array}{cccccc|l}
\text{abide} & \text{absence} & \text{absent} & \text{abundant} & \text{abyss} & \cdots & \\
\hline
1 & 2 & 3 & 1 & 1 & \cdots & \text{doc } 1 \\
1 & 0 & 2 & 0 & 0 & \cdots & \text{doc } 2 \\
0 & 1 & 1 & 0 & 0 & \cdots & \text{doc } 3 \\
0 & 2 & 0 & 0 & 0 & \cdots & \text{doc } 4 \\
0 & 3 & 1 & 0 & 0 & \cdots & \text{doc } 5 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}
$$

Word embedding, $\mathbf{V}$



Rise of Word Embedding Models

TITLE	CITED BY	YEAR
Distributed representations of words and phrases and their compositionality T Mikolov, I Sutskever, K Chen, GS Corrado, J Dean Neural information processing systems	16538	2013
Efficient estimation of word representations in vector space T Mikolov, K Chen, G Corrado, J Dean arXiv preprint arXiv:1301.3781	13414	2013
Glove: Global vectors for word representation J Pennington, R Socher, C Manning Proceedings of the 2014 conference on empirical methods in natural language ...	10790	2014

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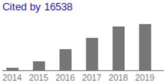

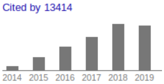

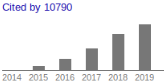

Invariance and identifiability issues for word embeddings R Carrington, K Bharath, S Preston Advances in Neural Information Processing Systems, 15114-15123	2019

Why word embeddings?

- Word embedding $\mathbf{V}$ encodes word meaning.
- Used for, and evaluated on, word tasks.
- Word similarity: "given word A, how similar is word B?"
- Word association: "A is to B as C is to what?", e.g. Paris is to France as Madrid is to ...?
- Task performance measured by $g(\text{data}, \mathbf{V})$.

Word embeddings as matrix factorisation

Simple word embedding model: take $\mathbf{V}$ as minimiser of

$$\|\mathbf{X} - \mathbf{U}\mathbf{V}\|^2 = \sum_{ij} \left(x_{ij} - \mathbf{u}_i^\top \mathbf{v}_j \right)^2$$

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Simple word embedding model: take **V** as minimiser of

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Solution for **V**: write SVD of $\mathbf{X} = \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}^\top$. Then $\|\mathbf{X} - \mathbf{X}_d\|$ is minimised by $\mathbf{X}_d = \mathbf{A}_d \boldsymbol{\Sigma}_d \mathbf{B}_d^T$, so take

$$\mathbf{U}^* = \mathbf{A}_d, \quad \mathbf{V}^* = \boldsymbol{\Sigma}_d \mathbf{B}_d^\top.$$

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... or $\mathbf{V}^* = \mathbf{\Sigma}_d^{1-\alpha} \mathbf{B}_d^\top$?

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$$\|\mathbf{X} - \mathbf{U}\mathbf{V}\|^2 = \sum_{ij} \left(x_{ij} - \mathbf{u}_i^\top \mathbf{v}_j \right)^2 = f(\mathbf{X}, \mathbf{U}\mathbf{V})$$

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Non-identifiability: $f(\mathbf{X}, \mathbf{U}\mathbf{V}) = f(\mathbf{X}, \mathbf{U}\mathbf{C}^{-1}\mathbf{C}\mathbf{V})$

Non-identifiability of different embedding models

$$\textbf{LSA:}\ \sum_{ij} \left(x_{ij} - \mathbf{u}_i^\top \mathbf{v}_j\right)^2$$

$$\textbf{word2vec:}\ \sum_{ij} \log\left(\sigma\left(\mathbf{u}_i^T \mathbf{v}_j\right)\right) + k \cdot \frac{\sum_l x_{il} \sum_m x_{mj}}{\sum_{ij} x_{ij}} \log\left(\sigma\left(-\mathbf{u}_i^T \mathbf{v}_j\right)\right)$$

$$\textbf{GloVe:}\ \sum_{ij} h\left(x_{ij}\right)\left(\mathbf{u}_i^T \mathbf{v}_j - h_1\left(x_{ij}\right)\right)^2$$

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Each is such that for any particular solution

$$\mathbf{V}^* = \arg\min_{\mathbf{V}} \ f(\mathbf{X}, \mathbf{UV})$$

a general solution set is

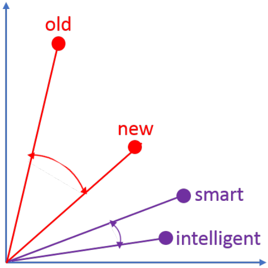
$$\{\mathbf{V} : \mathbf{V} = \mathbf{CV}^*, \mathbf{C} \in \mathrm{GL}(r)\}$$

Non-identifiability - implications? (1)

Word similarity task:

data

Word 1	Word 2	Human Score (y_i)	Embedding Score (z_i)
old	new	1.58	$\cos(\mathbf{v}_{\text{"old"}}, \mathbf{v}_{\text{"new"}})$
smart	intelligent	9.2	*etc*
hard	difficult	8.77	...
happy	cheerful	9.55	...
hard	easy	0.95	...
fast	rapid	8.75	...



$$\cos(\mathbf{v}_i, \mathbf{v}_j) = \mathbf{v}_i^\top \mathbf{v}_j / (\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|)$$

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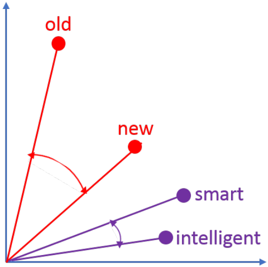

$$\cos(\mathbf{v}_i, \mathbf{v}_j) = \mathbf{v}_i^\top \mathbf{v}_j / \left(\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|\right)$$

...then measure “embedding performance” by

$$g(\text{data}, \mathbf{V}) = \text{corr}(\{y_i\}, \{z_i\})$$

Non-identifiability - implications? (2)

Word analogy task:

- Paris is to France as Madrid is to . . . ? Given $\mathbf{V}$ solve

$$\arg\max_i \left[\cos(\mathbf{v}_i, \mathbf{v}_{\text{“France”}}) - \cos(\mathbf{v}_i, \mathbf{v}_{\text{“Paris”}}) + \cos(\mathbf{v}_i, \mathbf{v}_{\text{“Madrid”}})\right]$$

- The data are a set of human-chosen analogies.
- Performance metric $g(\text{data}, \mathbf{V})$ is the proportion correct.

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For both similarity and analogy, g depends on $\mathbf{V}$ only via $\cos(\mathbf{v}_i, \mathbf{v}_j)$.

Hence $g(\text{data}, \mathbf{V}) = g(\text{data}, c\mathbf{Q}\mathbf{V}) \Rightarrow g$ is a shape function.

Mis-match of invariances

Training objective $f(\mathbf{X}, \mathbf{UV})$ invariant to $\mathbf{V} \mapsto \mathbf{CV}$.

Test objective $g(\text{data}, \mathbf{V})$ invariant to $\mathbf{V} \mapsto c\mathbf{QV}$

$$\begin{aligned} f\left(\boldsymbol{X}, \boldsymbol{UV}\right) &= f\left(\boldsymbol{X}, \boldsymbol{UC}^{-1}\boldsymbol{CV}\right), \qquad && \boldsymbol{C} \in \mathrm{GL}(r) \\ g\left(\boldsymbol{D}, \boldsymbol{V}\right) &= g\left(\boldsymbol{D}, c\boldsymbol{QV}\right), \qquad && \boldsymbol{Q} \in \mathrm{O}(d), c \in \mathbb{R} \end{aligned}$$

What is the set $\mathcal{F}_d \subset \mathrm{GL}(d)$ which leaves f invariant but not g?

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What is the set $\mathcal{F}_d \subset \mathrm{GL}(d)$ which leaves f invariant but not g?

Write $\mathcal{F}_d = \tilde{\mathcal{F}}_d - c\mathcal{I}$, where

- $\tilde{\mathcal{F}}_d = \mathrm{GL}(d) \backslash \mathrm{O}(d)$, and can be identified with $\mathrm{UT}(d)$, upper tringular matrices with +ve diag elements. (Intuition: QR decomposition of C)
- $c\mathcal{I} = \{cI_d : c \in \mathbb{R}\}$ is set of scale transformations
- dimension of $\mathcal{F}_d$ is $d(d-1)/2 - 1$.

Is the mis-match a problem?

- "When all methods are allowed to tune a similar set of hyperparameters their performance is largely comparable"[1]

[1] Levy, Goldberg, Dagan, *Trans. Assoc. Comput. Ling.*, 2015

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- Some hyperparameters index different elements of solution set f, chosen for performance in g, e.g. $\mathbf{V}^* = \mathbf{\Sigma}_d^{1-\alpha}\mathbf{B}_d^\top$
- f typically optimised by Monte Carlo (different solns explained as local optima - but also due to non-identifiabilty) then soln chosen for g.

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- f typically optimised by Monte Carlo (different solns explained as local optima - but also due to non-identifiabilty) then soln chosen for g.
- Both are (implicitly) *supervised* approaches.

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Resolving mis-match

$$\arg\min_{\mathbf{V}} \; f(\mathbf{X}, \mathbf{UV})$$

Identifying a solution unique up to orthogonal transformations:

- Impose constraint $\mathbf{V}\mathbf{V}^\top = \mathbf{I}$, then for any solution $\mathbf{V}^*$ any other solution $\mathbf{C}\mathbf{V}^*$ for $\mathbf{C} \in \mathrm{GL(d)}$ satisfies $g(\mathrm{data}, \mathbf{C}\mathbf{V}^*) = g(\mathrm{data}, \mathbf{V}^*)$.

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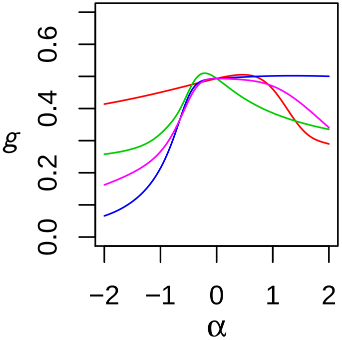
Identifying a unique solution:

- Additionally impose: (i) $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$, (ii) $\mathrm{diag}(\mathbf{U}^\top\mathbf{U})$ decreasing, (iii) positive first non-zero elements of each col of $\mathbf{U}$.

Sensitivity to particular solution

$$\mathbf{V} = \mathbf{\Lambda}^{\alpha} \mathbf{V}^{*}$$



$\mathbf{\Lambda} = \mathbf{\Sigma}_d$ $\quad (\mathbf{\Lambda})_i \sim |N(0,1)|$

$(\mathbf{\Lambda})_i = i$ $\quad (\mathbf{\Lambda})_i \sim U(0,1)$

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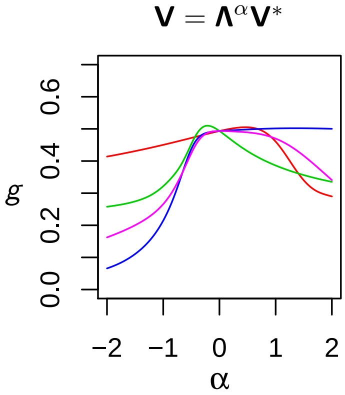

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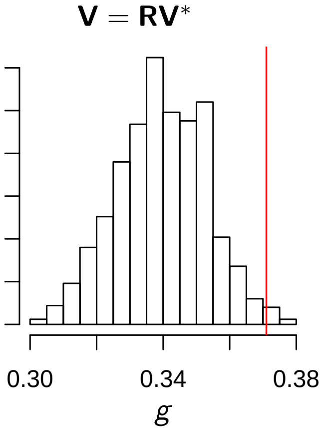
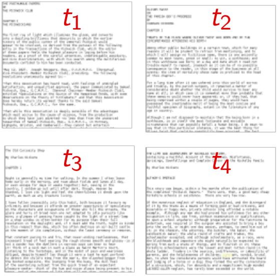


$(\mathbf{R})_{ij} \sim |N(0,1)|;\ j \geq i$

Outlook: dynamic embedding

Text data



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\begin{array}{c}
\begin{array}{cccccc} \text{abide} & \text{absence} & \text{absent} & \text{abundant} & \text{abyss} & \cdots \end{array} \\
\begin{pmatrix}
1 & 2 & 3 & 1 & 1 & \cdots \\
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\end{array}
\begin{array}{l}
\\
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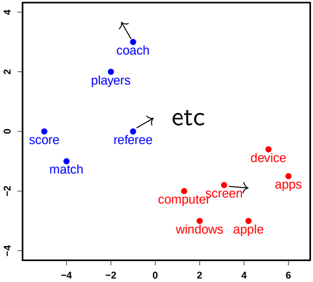
\text{doc } 4 \quad t_4 \\
\text{doc } 5 \quad t_5 \\
\end{array}
$$

Word embedding, $\mathbf{V}(t)$



Acknowledgements

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